AD-A060 111 MINNESOTA UNIV MINNEAPOLIS F/G 12/1

WAVE PROPAGATION IN NONLINEAR PARABOLIC PROBLEMS AND OTHER PROB--ETC(U)

SEP 78 J B MCLEOD, J SERRIN DAAG29-77-G-0157

UNCLASSIFIED ARO-14997.1-M NL

| OF |

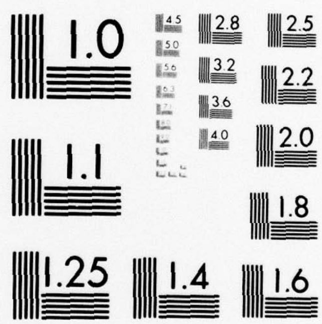
AD A060111

END

DATE FILMED

12 78

DDC



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

LEVEL

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(18) ARO (19) 14997.1-M

ADA060111

DDC FILE COPY

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
Final Report

2. GOVT ACCESSION NO.
(5)

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
(6) Wave Propogation in Nonlinear Parabolic Problems and Other Problems,

5. TYPE OF REPORT & PERIOD COVERED
(9) Final report, 16 SEP 77-15 JUN 78,

6. PERFORMING ORG. REPORT NUMBER
Final report

7. AUTHOR(s)
(10) J.B./McLeod
J./Serrin

8. CONTRACT OR GRANT NUMBER(s)
(15) DAAG 29-77-G-0157

9. PERFORMING ORGANIZATION NAME AND ADDRESS
U. of Minnesota
MINNEAPOLIS, MN 55455

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
-

11. CONTROLLING OFFICE NAME AND ADDRESS
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709

(11) 25

12. REPORT DATE
Sep[redacted], 1978

13. NUMBER OF PAGES
9

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
(12) 11 p.

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
Unlimited

18. SUPPLEMENTARY NOTES
The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Final grant report, Wave Propogation, Parabolic Problems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Final grant report

78 10 10 063

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified 233 500 VB
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

WAVE PROPOGATION IN NONLINEAR
PARABOLIC PROBLEMS AND OTHER PROBLEMS

Final Report

J.B. McLeod
J. Serrin

September 25, 1978

U. S. ARMY RESEARCH OFFICE

DAAG 29-77-G-0157

University of Minnesota

FINAL REPORT

I. Problems studied

The following problems have been studied:

CONTENTS:

1. Convergence to travelling waves of solutions of nonlinear diffusion equations.
2. Stability of travelling pulses of systems of reaction-diffusion equations.
3. The relation between the inverse scattering method for nonlinear wave equations and connection problems for certain nonlinear ordinary differential equations, and
4. Stokes' conjecture concerning the existence of a wave of greatest height and the angle of its slope.

A summary of the results obtained follows in the succeeding sections.

1. Convergence to travelling waves of solutions of nonlinear diffusion equations

It was indicated in the proposal for the current grant that Fife and McLeod had announced results in [1] which effectively answer the problem in the following case.

Let us take the equation

$$(1.1) \qquad u_t - u_{xx} - f(u) = 0 \qquad (-\infty < x < \infty, t > 0)$$

with the initial value

$$u(x,0) = \phi(x) .$$

Suppose that $f \in C'[0,1]$, with

$$(1.2) \qquad f(0) = f(1) = 0$$

and

$$(1.3) \qquad f(u) < 0 \text{ for } 0 < u < \alpha < 1 \text{ , } f(u) > 0 \text{ for } \alpha < u < 1 \text{ ,}$$

$$(1.4) \qquad f'(0) < 0 \text{ , } f'(1) < 0 \text{ .}$$

There is then one and (except for translation) only one wave $U(x-ct)$ with $U(-\infty) = 0$, $U(+\infty) = 1$, and the solution $u(x,t)$ will converge exponentially fast to a specific translation of this travelling wave if the initial function ϕ satisfies $0 \leq \phi \leq 1$ and

$$\overline{\lim_{x \to -\infty}} \phi(x) < \alpha \text{ , } \underline{\lim_{x \to +\infty}} \phi(x) > \alpha \text{ .}$$

There are also results which cover the cases where, for example, ϕ is of compact support or where f, while still satisfying $f'(0) < 0$ and $f'(1) < 0$, has more internal zeros in $(0,1)$ than just α.

These results have now been placed in final form, and have appeared in [2].

The method used in [2] is to apply the maximum principle in association with the equation (1.1). An alternative approach, however, is to transform the problem to phase plane variables, so that the independent variables are u and t and the dependent variable $p = u_x$. In the first instance this is possible only if the initial function ϕ in monotone (which implies that $u(x,t)$ is monotonic in x for all t), but this restriction can be avoided. By applying the maximum principle to the diffusion equation transformed in terms of the phase plane variables, the results of [2] can be extended effectively to any f for which $f(0) = f(1) = 0$. A first report on this has been prepared in [3] and a record is in preparation. In this area the objectives cited in the original proposal have been fully realized.

2. Stability of travelling pulses of systems of reaction-diffusion equations

Work here has been restricted to the Fitzhugh-Nagumo system

$$(2.1)\qquad \begin{cases} u_t = u_{xx} + f(u) - w \,, \\ \\ w_t = \epsilon u \quad , \end{cases}$$

where f satisfies the conditions (1.2-4) and ϵ is a small positive parameter. Expanding on work by Hastings [4], we have shown that there exist two travelling pulses for this system, i.e. non-trivial solutions $(u(x,t),w(x,t)) = (U(x-ct),W(x-ct))$ with

$$U(\pm\infty) = 0 \,, \quad W(\pm\infty) = 0 \,,$$

and that one has a slow speed for small ϵ, while the other has a speed which does not tend to zero as $\epsilon \downarrow 0$. We have investigated the behavior of these pulses and so have been able to treat (2.1) as a singular perturbation of (1.1) and show that the slow pulse is unstable and the other pulse stable. Two reports on this are in preparation and again the objectives of the original proposal have been fully realized.

3. The relation between the inverse scattering method for nonlinear wave equations and connection problems for certain nonlinear ordinary differential equations

The equation

$$y'' - xy = y|y|^{\alpha} \quad , \tag{3.1}$$

where α is a positive constant, arises in plasma physics, and for $\alpha = 2$, the most important case, it is known as the second Painlevé transcendent and yields similarity solutions to the well known Korteweg-de Vries equation. If solutions are sought which satisfy the boundary conditions

$$y(\infty) = 0 \quad ,$$
$$y(x) \sim (-\tfrac{1}{2}x)^{1/\alpha} \quad \text{as} \quad x \to -\infty \quad ,$$

then it can be shown that there exists a unique such solution, and a corresponding constant $k(\alpha)$ such that as $x \to +\infty$,

$$y(x) \sim k(\alpha) Ai(x) \quad ,$$

$Ai(x)$ being Aroy's function. To determine $k(\alpha)$ is a connection problem for the nonlinear equation (3.1), and would expect that obtaining a value for $k(\alpha)$ by analytic means would be impossible. However, numerical calculations had indicated strongly that, in the case $\alpha = 2$, we have the surprising result that $k(2) = 1$, and we have been able to establish this analytically by exploiting both the relationship between (3.1) and the Korteweg-de Vries equation, and also the inverse scattering method for solutions of the Korteweg-de Vries equation.

It seems clear that these ideas extend to other nonlinear wave equations which can be solved by inverse scattering and to the associated ordinary differential equations satisfied by their similarity solutions, and there is also a close link between these ordinary differential equations and the class of Poincaré transcendents. One report on this work has appeared in [5], and further work is in progress.

4. Stokes' conjecture concerning the existence of a wave of greatest height and the angle of its slope

Consider the problem of a wave of constant periodic form moving with constant velocity on the surface of a non-viscous fluid which is either of infinite depth or on a horizontal bottom. This was reduced by Nekrasov [6] by complex variable methods to the discussion of the integral equation

$$(4.1) \qquad \phi(s) = \frac{1}{3\pi}\int_0^{\pi} \frac{\sin\phi(t)}{\mu^{-1} + \int_0^t \sin\phi(u)\,du} \log\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| dt$$

if the depth is infinite, and to another similar equation of the depth is finite. Here $\phi(s)$ is the angle between the wave surface and the horizontal at the point or the surface corresponding to the independent variable s, and the crest of the wave occurs at $s = 0$ and the length at $s = \tilde{n}$. The constant μ is given by

$$\mu = \frac{3g\lambda c}{2\pi Q^3} \quad ,$$

where g is the acceleration due gravity, λ the wave-length of the periodic wave, c the speed at which the wave form is progressing, and φ the speed of particles at the crest of the wave. The case $\mu = \infty(\varphi=0)$ thus corresponds to a stagnation point at the wave crest and is also the case in which, for given c, the wave reaches the greatest height above mean level. In 1880 Stokes [7] conjectured that there exists a wave in this limiting case, but that it is peaked instead of smooth-crested, and that at the peak the wave makes an angle $\frac{1}{6}x$ with the horizontal i.e.

(4.2) $$\lim_{s \downarrow 0} \phi(s) = \frac{1}{6}\pi \quad .$$

Two further conjectures were made by Krasovskii [8] in an investigation of the existence of solutions of (4.1) for finite μ . The key role played by an inequality $|\phi| < \frac{1}{6}\pi$ in his own work led him to hypothesis in essence that

(1) as $\sup_{s \in [0,\pi]} |\phi(s)| \uparrow \frac{1}{6}\pi$, the solution tends to Stokes' limit solution, and

(2) there is no solution for which $\sup_{s \in [0,\pi]} |\phi(s)| > \frac{1}{6}\pi$.

Toland [9] has recently proved that there does indeed exist a solution in the limit case, but his proof is complicated. Further, doubt has been thrown or the validity of Krasovskii's conjectures by some numerical work by Longuet-Higgins and Fox [10]. We have succeeded in producing a simpler proof of existence than Toland's, and have also established by rigorous analysis that Krasovskii's conjectures are indeed false, in that, for μ sufficiently large it must be the case that $\sup|\phi| > \frac{1}{6}\pi$. This does not invalidate Stokes' original conjecture (4.2), for $|\varphi|$ exceeds $\frac{1}{6}\pi$ only is a boundary layer which disappears in the limit as $\mu \to \infty$; but it does emphasize that the passage to the limit is a delicate one, and (4.2) itself, although presumably true, remains unproved.

II. Bibliography

1. P.C. Fife and J.B. McLeod, Bull. American Math. Soc. 81 (1975), 1076-1078.

2. P.C. Fife and J.B. McLeod, Arch. Rational Mech. Anal. 65 (1977), 335-361.

3. P.C. Fife and J.B. McLeod, Technical Summary Report, Mathematics Research Center, University of Wisconsin.

4. S.P. Hastings, Quart. J. Math. (Oxford) 27 (1976), 123-134.

5. S.P. Hastings and J.B. McLeod, Technical Summary Report No. 1861, Mathematics Research Center, University of Wisconsin.

6. A.I. Nekrasov, Technical Summary Report No. 873, Mathematics Research Center, University of Wisconsin.

7. G.G. Stokes, Mathematical and Physical Papers (Vol. I, C.U.P., 1880).

8. Yu. P. Krasovskii, U.S.S.R. Comp. Math. and Math. Phys. 1 (1962), 996-1018.

9. J.F. Toland, Report No. 87, Fluid Mechanics Research Institute, University of Essex.

10. M.S. Longuet-Higgins and M.J.H. Fox, J. Fluid Mech. 80 (1977), 721-741.